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Horizon entropy and higher curvature equations of state

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The Clausius relation between entropy change and heat flux has previously been used to derive Einstein's field equations as an equation of state. In that derivation the entropy is proportional to the area of a local causal horizon, and the heat is the energy flux across the horizon, defined relative to an approximate boost Killing vector. We examine here whether a similar derivation can be given for extensions beyond Einstein gravity to include higher derivative and higher curvature terms. We review previous proposals which, in our opinion, are problematic or incomplete. Refining one of these, we assume that the horizon entropy depends on an approximate local Killing vector in a way that mimics the diffeomorphism Noether charge that yields the entropy of a stationary black hole. We show how this can be made to work if various restrictions are imposed on the nature of the horizon slices and the approximate Killing vector. Also, an integrability condition on the assumed horizon entropy density must hold. This can yield field equations of a Lagrangian constructed algebraically from the metric and Riemann tensor, but appears unlikely to allow for derivatives of curvature in the Lagrangian.

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I. INTRODUCTION

The notion of horizon entropy and thermodynamics was first discovered for black holes in general relativity (GR) and quickly generalized to other sorts of horizons. The origin of this thermodynamic behavior can be traced to local physics, and in a sense arises from the nature of the vacuum. This led to the observation that the Einstein equation can be derived as an equation of state for local causal horizons in the neighborhood of a point 'p' in spacetime, by imposing the Clausius relation between their entropy change and the energy flux across them [1]. In this paper we examine approaches to generalizing this equation of state derivation to allow for higher derivative contributions to the entropy and field equations. The Einstein-Hilbert Lagrangian is only the lowest order term (other than the cosmological constant) in a derivative expansion of generally covariant actions for a metric theory, and the presence of higher derivative terms is presumably inevitable. Several approaches to including higher derivative terms in the entropy and equation of state have been tried. In our view none have fully succeeded, except in the case of $\mathcal{L}(R)$ theories. Those theories are special however, since they are trivially related to general relativity coupled to a scalar field by a field dependent conformal rescaling of the metric. Here we will explain the problems that arise with previous proposals, and propose a solution that adopts aspects of some of the proposals.

The solution differs in several ways from the original derivation for GR, among which are: (i) the entropy is compared on two horizon slices that share a common boundary, (ii) the bifurcation surface lies to the past of the terminal point p and, (iii) the entropy depends on the approximate Killing vector. In particular, it has the same dependence on the approximate Killing vector as have the Noether charges associated with a Lagrangian, in analogy with the Wald entropy [2, 3] for stationary black holes. Such dependence will be referred to as "Noetheresque", but by itself does not make the entropy a Noether charge. It makes some sense that the entropy depends on the approximate Killing vector, because the latter determines the notion of stationarity and enters the definition of the heat flux. However, at present we can offer no statistical interpretation for this form of the entropy.

The equation of state we can derive is consistent with local energy-momentum conservation only if the leading order term in the entropy satisfies an integrability condition. This condition is satisfied if the entropy arises from variation of a generally covariant function with respect to curvature. In other words, the entropy coincides with a Noether charge associated with a (particular type of) gravitational Lagrangian. The need for such an integrability condition was anticipated in Ref. [1], since it was known that the entropy of stationary black hole horizons has this form[3–5]. We have not been able to ascertain whether this is the only way to satisfy the integrability condition, and in particular whether field equations for Lagrangians involving derivatives of curvature can be obtained.

Besides the lack of a statistical interpretation, and the dependence of the entropy on the local Killing vector, a strange feature of this approach is that in the case of GR the entropy on a general horizon slice differs from the area at the same order as the relevant area changes in the Clausius relation. So this approach seems to lose contact with some of the original statistical motivation for the local Clausius relation. On the other hand, it is

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quite analogous to the first law of black hole mechanics in generalized gravity theories. Therefore it is not clear to us whether this approach is purely formal, or maintains some significance as true thermodynamics of spacetime.

II. THE EINSTEIN EQUATION OF STATE

We begin with a review of the derivation of the Einstein equation as an equation of state, which emerges from black hole thermodynamics as follows. General relativity and quantum field theory in a black hole background imply the so-called "first law of black hole thermodynamics" [6–8],

$$dM - \Omega_H dJ = T_H dS_{BH}, \tag{1}$$

where M is the black hole mass, Ω_H the angular velocity of the horizon, J the angular momentum, $T_H = \hbar\kappa/2\pi$ with surface gravity κ is the Hawking temperature, and $S_{BH} = A/(4\hbar G)$ with horizon area A is the Bekenstein-Hawking entropy. This relation can be viewed as a comparison between two stationary black holes, but it also holds for small, slowly time-dependent changes of a single black hole. Its validity in that setting hinges on the fact that the evolution of horizon area is governed by spacetime curvature, which in turn is linked via Einstein's equation to the energy flux. Specifically, it relies on the relation $R_{ab}k^ak^b = 8\pi GT_{ab}k^ak^b$, where R_{ab} is the Ricci tensor, T_{ab} is the energy momentum tensor of matter, and k^a is a 4-vector tangent to the horizon generating null geodesics.

The term "first law" is actually a misnomer for (1). In thermodynamics that name refers to energy conservation, $dU = \delta Q + \delta W$, where dU is the internal energy change, δQ is the heat flow into and δW is the work done on the system. Instead, the thermodynamic nature of (1) is the Clausius relation

$$\delta Q = T\delta S \tag{2}$$

between the heat flow and the entropy change. In the black hole context, energy that flows across the horizon is, in effect, "heat", since after crossing the horizon its microscopic nature is effaced for an outside observer. Since the horizon is a causal barrier, it is a "perfect dissipator" [9]. The heat flux is

$$\delta Q = dM - \Omega_H dJ = \int_{\mathcal{H}} (-T_{ab}\chi^a) dH^b, \tag{3}$$

where the integral is over the horizon and $\chi^a = \partial_t^a + \Omega_H \partial_\phi^a$ is the horizon-generating Killing vector.[1] That is, the heat is the energy flux conjugate to the spacetime symmetry that translates along the horizon generators. The work term δW in the usual first law of thermodynamics has no analog in the relation (1). Although "first law" is not an appropriate name, we will use it here since it is entirely standard terminology.

Close to a black hole horizon the Hawking temperature becomes the Unruh temperature $\hbar a/2\pi$ for stationary, uniformly accelerated observers, and the Clausius relation takes on a local form (how local depends on how slowly the changes take place) whose validity requires that the Einstein equation hold, as mentioned above. Moreover, there is good reason to believe this applies not just to black holes but to any causal horizon. (See e.g. [10] for a review of the arguments, and [11] for further discussion and clarification.)

The idea of Ref. [1] (see also [12] for a slight reformulation) was that, conversely, the Einstein equation can be derived, as an equation of state, by requiring that the Clausius relation hold for the entropy of sufficiently small patches of all local causal horizons (LCHs) in spacetime. Any such horizon H is defined as the boundary of the past of a patch of $(D-2)$-dimensional spacelike surface, where D is the spacetime dimension. (LCHs will be defined precisely in what follows; for now it suffices to remain somewhat vague.) In that derivation the heat is taken as the boost energy flux, defined with respect to an approximate local boost Killing vector field ξ^a as

$$\delta Q = \int_H (-T_{ab}\xi^a) dH^b, \tag{4}$$

and the temperature is taken as the Unruh boost temperature $\hbar/2\pi$. This is natural since the Minkowski vacuum state of quantum fields is thermal with respect to the boost Hamiltonian at this temperature, and any state looks like the Minkowski vacuum at sufficiently short distances. The entropy is taken to be A/l_0^2, where l_0 is some UV length scale. (A compelling case can be made that the origin of horizon entropy is quantum entanglement of vacuum correlations across the horizon. Since this is dominated by UV degrees of freedom, yet finite, dimensional analysis suggests that the leading order contribution should scale in this way.) The change in horizon area in a small neighborhood of a point p can be related to the Ricci tensor via the Raychaudhuri equation. If the horizon is chosen so its expansion and shear vanish at p, and ξ^a is chosen so it too vanishes at p, the Clausius relation is then seen to require $T_{ab}k^ak^b = (\hbar/(2\pi l_0^2))R_{ab}k^ak^b$ at every point in spacetime and for all null vectors k^a. Together with energy conservation $\nabla^a T_{ab} = 0$ this implies the Einstein equation

$$R_{ab} - \tfrac{1}{2}Rg_{ab} - \Lambda g_{ab} = 8\pi G_0 T_{ab}, \tag{5}$$

where the value of Newton's constant is determined by the entropy density $1/l_0^2$ to be $G_0 = l_0^2/4\hbar$, and Λ is an undetermined cosmological constant.

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[1] We (unfortunately) use spacetime signature $(-+\cdots+)$. The horizon integration measure is $dH_b = -k_b dV dA$, where V is the affine parameter along the horizon generators, $k^b = (\partial_V)^b$ is tangent to the generators, and dA is the area element of a constant V horizon slice.

III. PREVIOUS APPROACHES TO INCLUDING HIGHER DERIVATIVES

There have been a number of attempts to include higher derivative terms in the equation of state derived from causal horizon entropy; all these attempts can be divided into two broad classes: The first deals with a specific theory, $L(R)$ gravity (see Refs. [12–14]), while the second class studies a more general case(e.g. Refs. [15, 16]). The $L(R)$ gravity, where the Lagrangian depends only on the Ricci scalar, is equivalent to general relativity with an auxiliary scalar field [17, 18] and is therefore the simplest generalization possible. Even in that case, the derivation of the field equation from thermodynamics of local horizon meets considerable obstacles.

Ref. [12] considered the case when the horizon entropy density is proportional to an arbitrary function $f(R)$ of the spacetime Ricci scalar R. As was done in the derivation of the Einstein equation, the approximate Killing vector ξ is chosen to vanish at a terminal point p on the horizon. The Clausius relation equates the entropy change to the heat flux δQ, which is the flux of boost energy current $-T^{ab}\xi_b$ across the horizon. Since ξ vanishes at the terminal point, the rate of change of entropy with respect to the affine parameter must also vanish there. However, at a generic spacetime point p the gradient $\nabla_a R$ is non-vanishing, so the horizon entropy will have a nonvanishing rate of change unless the change of R is balanced by a change of horizon area. Thus it is necessary to adjust the terminal surface of the horizon so that the expansion θ of the horizon generators at the equilibrium point has the nonzero value $\theta_p = -\dot{f}/f$. The Clausius relation involves a θ_p^2 term via the Raychaudhuri equation, and one ends up with a field equation that is inconsistent with matter energy conservation unless this term is identified as internal entropy production due to a bulk viscosity proportional to $f(R)$ and added to the entropy balance law. Hence, instead of equilibrium thermodynamics, a non-equilibrium approach is needed. The resulting field equation is the one that follows from the Lagrangian L, where $f = dL/dR$. That is, L is the Lagrangian for which $f(R)$ is the Wald entropy [3]. (The properties of Wald entropy will be briefly reviewed in section VII.)

In Ref. [13], this entropy production term is interpreted as a separate contribution to the heat flux from the additional scalar degree of freedom present in $L(R)$ gravity. Then it is possible to derive the field equations using only reversible thermodynamics as in the case of general relativity. Although such an interpretation may offer a possible understanding of entropy production terms for the specific case of $L(R)$ gravity, it is unclear how to generalize that for a broader class of theories.

In Ref. [14] it was proposed that the need for internal entropy production could be eliminated by adopting the instantaneous boost invariant (IBI) prescription for dynamical horizon entropy proposed in [3]. This prescription goes as follows. In the neighborhood of a spatial slice Σ of a causal horizon one constructs, by a unique recipe of dropping selected terms in a Taylor expansion of the true metric in an adapted coordinate system, a new spacetime metric that has an exact boost Killing field for which Σ is a fixed point set. The entropy density of the slice Σ was taken in [14] to be a scalar formed from the IBI metric associated to Σ. In particular the case with entropy density $f(R)$ was discussed, but it is not clear to us that all contributions to the change in this entropy were taken into account. The difficulty arises because the IBI metric changes with the horizon slice, so the R on each slice is defined with respect to a different metric. Also, even if the field equation arrived at in [14] were correct, it refers to the curvature of the IBI metric, and it remains unclear how this is related to the curvature of the original metric.

Next we would like to discuss attempts to obtain the field equation from local horizon thermodynamics beyond $L(R)$ gravity. Ref. [15] starts from the Wald entropy formula [3] for a stationary black hole, but applied to a local horizon. A formula for the entropy of a $[(D-1)$-dimensional$]$ patch of a local horizon is written as a surface integral over the boundary of the patch. It is unclear to us why an entropy should be assigned to a $(D-1)$-patch rather than only to a $(D-2)$-slice of the horizon. Moreover, in the case of a stationary black hole horizon this integral vanishes, whereas the entropy of the horizon surely does not vanish. Hence this quantity cannot be interpreted as the entropy of the horizon. The variation of this entropy is then considered, and a formula is written in terms of an integral of a covariant directional derivative of part of the integrand of the previously mentioned integral. It is unclear to us why this expression should be interpreted as the change of entropy. A similar formula is written for the boost energy flux across the horizon. Also, the binormal on horizon cross sections is identified with the covariant derivative of the approximate Killing vector. Whereas this can certainly be satisfied to lowest order, given the small changes involved in the Clausius relation it would be necessary to check whether neglecting the differences at subleading order is justified. Another issue is that the approximate Killing vector ξ^a in these calculations is treated as if it satisfies the Killing identity $\nabla_a\nabla_b\xi_c = R^d{}_{abc}\xi_d$ exactly. Although this is not possible in general, we will show in the present paper that the identity can be satisfied to the required order, provided one restricts to a narrow neighborhood of a particular horizon generator.

In Ref. [16] the proposal is to define the entropy of horizon slices as the integral of a Noether potential formally identical to a potential introduced in [3], except that in the former case no matter fields are included in the Lagrangian with which the potential is associated. Using Stokes' theorem, the entropy change between two slices of the horizon is then expressed as an integral of the correspondig Noether current over the enclosed

horizon patch. However, since slices of a LCH have boundaries, the entropy change also involves a surface integral over the null outer boundary, unless the two slices have their boundary in common. This contribution was missed in Ref. [16]. In addition, as in Ref. [15], the Killing identity was used without justification at an order it cannot be expected to hold in a generic spacetime. Thus this derivation too is not complete. The present paper starts from a similar but more general form for the entropy, and fills these gaps in the argument by studying slices that do have their boundary in common and thus enclose a compact patch of the horizon.

Finally we note that Refs. [19, 20] (see also [21] and references therein) discuss a thermodynamic interpretation of gravitational field equations beyond general relativity in terms of a local entropy balance law. A matter entropy flux across the horizon is associated with a small spatial volume and is related to the boost energy by the Clausius relation. This is then equated to a gravitational entropy change in the volume, which is constructed from the Noether current associated with a Lagrangian $L[g_{ab}, R^a{}_{bcd}]$. This balance law at a point is then shown to imply that the gravitational field equations hold, thus giving a thermodynamical interpretation of those equations. The approach we will pursue in this paper uses similar ingredients, but we have different objectives. We start by assigning a horizon entropy functional to a small but finite sized horizon slice, and investigate what properties it must have if it is to satisfy the Clausius relation with the matter boost energy flux.

IV. IMPOSSIBILITY OF GENERALIZING THE NON-EQUILIBRIUM APPROACH

A natural question to ask is whether the non-equilibrium approach of Ref. [12] can be extended beyond theories for which the horizon entropy density s is a function only of the Ricci scalar. In this section it will be shown that the non-equilibrium approach of Ref. [12] cannot be extended beyond the results found there. For this discussion, we adopt the geometric set up of [1, 12] and we assume that the entropy associated with the LCH takes the form

$$S = \int s \; dA, \tag{6}$$

where s is some arbitrary scalar function and the integration is over a $(D-2)$-dimensional spacelike slice of the horizon with 'area' element dA. The change in entropy from one horizon slice to another is

$$\delta S = \int \left(\frac{ds}{d\lambda} + \theta s \right) d\lambda \; dA, \tag{7}$$

where λ is an affine parameter on the local horizon generators and $\theta = d(\ln dA)/d\lambda$ is the expansion of the generators.

The Clausius relation asserts that the entropy change δS is equal to $\delta Q/T$, where $T = \hbar/2\pi$ is the boost temperature, δQ is the heat flux (4) through the horizon and ξ^a is the approximate Killing vector which vanishes at the final equilibrium point p. Since the heat integrand vanishes at p, the Clausius relation can hold only if the δS integrand also vanishes there. This requires that the expansion at p is nonzero, to wit

$$\theta_p = -\frac{1}{s} \left. \frac{ds}{d\lambda} \right|_p . \tag{8}$$

In the case of GR s is a constant, so this condition states that the horizon must have vanishing expansion at p. The Raychaudhuri equation can be used to find $\theta(\lambda)$. If the horizon shear does not vanish at p we must add internal entropy production due to shear viscosity to the Clausius relation [12]. Assuming the shear at p vanishes, we have $\theta = \theta_p + \lambda(-\theta_p^2/(D-2) - R_{ab}k^a k^b) + O(\lambda^2)$, where k^a is the affine horizon tangent vector $(\mathrm{d}/\mathrm{d}\lambda)^a$, and $\lambda_p = 0$.

Now we require that the Clausius relation hold for all local horizons through p, i.e. for all k^a. If the entropy density is independent of k^a, as would be the case if it is a spacetime scalar, then we obtain the tensorial equation

$$sR_{ab} - \nabla_a \nabla_b s + \frac{D-1}{D-2} s^{-1} s_{,a}\, s_{,b} + \Psi g_{ab} = \frac{2\pi}{\eta} T_{ab}, \tag{9}$$

where Ψ is some scalar function. This function can be determined by imposing energy conservation, i.e. $\nabla^a T_{ab} = 0$, which yields the condition

$$\Psi_{,a} = -\frac{1}{2} s R_{,a} + \partial_a \Box s - \left(\frac{D-1}{D-2} s^{-1} s_{,a}\, s_{,b} \right)^{;b} . \tag{10}$$

Since $\Psi_{,a}$ is the gradient of a scalar, a solution exists only if the right hand side is also the gradient of a scalar.

In the case of GR s is a constant, so $\Psi = -sR/2$ is the unique solution up to a constant (the cosmological constant). If $s = s(R)$ is a function only of the Ricci scalar, then the first term on the right hand side of (10) is a gradient, but the last term is not. In the non-equilibrium approach of Ref. [12] the Clausius relation is replaced by an entropy balance relation $\delta S = \delta Q/T + \delta S_i$, where δS_i is an internal entropy production term due to bulk viscosity, which is proportional to the square of the expansion and which cancels the last term in (10). This yields an equation of state that coincides with the field equations for $L(R)$ gravity. For more general spacetime scalar entropy densities, e.g. $s(R_{abcd})$, even the first term is not a gradient, so there is apparently no way to satisfy an entropy balance law, even allowing for internal entropy production. Hence, the methods devolved in [1] and [12] can not be generalized beyond $L(R)$.

The situation is even more problematic if the entropy density is not a spacetime scalar but, for example, is constructed from the intrinsic curvature of the $(D-2)$-dimensional horizon slice, as in Lovelock gravities [22]. In that case, we do not even have a tensorial relationship like Eq.(9). The Raychaudhuri equation then does

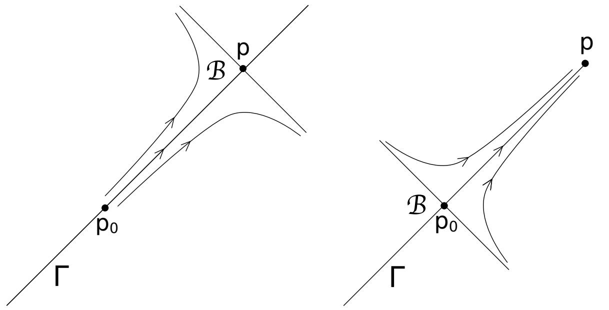

Figure 1: Causal spacetime diagram of the local causal horizon (LCH) and Killing field in the old (left) and new (right) setups. Each point in the diagram represents a patch of spacelike 2-surface. The boundary of the past of a patch of 2-surface through p defines the LCH. The arrows indicate the flow lines of the local Killing field ξ^a, which is of third order on the bifurcation surface $\mathcal{B}$, and which vanishes at p (left) or p_0 (right). The central horizon generator Γ runs from p_0 to p. The "heat" is the boost energy flux across the horizon. In the new setup the Killing vector is timelike within the horizon and spacelike beyond, while the old setup is the opposite.

not seem to be of any help at all, so a very different approach is needed. In this paper we will present such an alternative approach, which involves an entropy density that depends explicitly on the Killing vector, a different choice of horizon slices, and the application of Stokes' theorem.

V. NEW CHOICE OF APPROXIMATE KILLING VECTOR

It turns out that the new approach to the thermodynamic derivation of field equations, even in the case of general relativity, calls for a small but essential adjustment in the choice of approximate Killing vector. This adjustment amounts to locating the bifurcation surface[2] at a time earlier than the terminal point, rather than being coincident with the terminal point as originally formulated in Ref. [1]. The change also makes the application of the Clausius relation more closely analogous to the first law of black hole mechanics [23] and resolves an uncomfortable aspect of the earlier derivation. In this section we explain the thermodynamic motivation for the new choice of approximate Killing vector, and show how the original derivation of Ref. [1] is modified by this choice. The role in the new approach will be explained in Sec. VIII.

The role of the approximate Killing field ξ^a in the approach of Ref. [1] is to define the heat in the Clausius relation (4). The bifurcation surface of this Killing field was previously taken to coincide with the future boundary through p, whereas in the first law of global horizon mechanics the bifurcation surface lies to the *past* of the perturbation. Moreover, the approximate Killing field was spacelike rather than timelike in the region outside the horizon, so it was necessary to think of the heat as going into the reservoir *behind* the horizon (see Fig. 1).[3] However, this reservoir is not observable on the outside, so should play no role in the outside thermodynamics. It seems much more satisfactory to place the bifurcation surface to the past, so that the Killing field is timelike outside the horizon, and the reservoir can be thought of in direct analogy with the thermal atmosphere of a black hole, below the stretched horizon.[4]

It appears at first that modifying the location of the bifurcation point of the Killing vector will change the heat flux and, in the original approach of Ref. [1], ruin the derivation of the Einstein equation from the Clausius relation. However, that is not what happens. The old Killing vector on the generator through p (see Fig. 1) was $\xi_b^{\rm old} = -\lambda k_b$, where λ is the affine parameter that vanishes at p. The new Killing vector vanishes instead at p_0 and is given by $\xi_b = (\lambda - \lambda_0)k_b$, where λ_0 is the value of λ at p_0. The corresponding boost energy currents are thus

$$\begin{aligned} -T^{ab}\xi_b^{\rm old} &= \lambda T^{ab}k_b && (11)\\ -T^{ab}\xi_b &= (\lambda_0-\lambda)T^{ab}k_b && (12)\end{aligned}$$

Although these differ, their integrals from the bifurcation point to p are the same, since

$$\int_{\lambda_0}^{0}\lambda\, d\lambda = -\lambda_0^2/2 = \int_{\lambda_0}^{0}(\lambda_0-\lambda)\, d\lambda. \qquad (13)$$

With the heat defined using ξ^a, applying the Clausius relation to the horizon interval $[\lambda_0, 0]$ in the limit $\lambda_0 \to 0$ thus yields the field equations as described above. This corresponds to a transition between a stationary state at $\lambda = \lambda_0$ (where ξ^a vanishes) and one at $\lambda = 0$ (where the expansion and shear vanish). In analogy to the physical process version of the first law of black hole mechanics, the Clausius relation holds only when applied to this entire interval.

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[2] In the current context, the term "bifurcation surface" is used somewhat loosely, mainly to fix attention to the analogous role played by the true bifurcation surface of a quasi-stationary black hole in the physical process version of the first law [23]. What we mean is a slice on which the components of the approximate Kiling vector are third order in an appropriate coordinate system (see Sec. VI D). Accordingly, the approximate Killing vector vanishes at the center of the bifurcation surface (see Fig. 1).

[3] We use the term "outside" to refer to the region accessible to observers to the past of p, like the observers outside a black hole horizon. For a cosmological horizon, the standard terminology would be opposite: outside a cosmological horizon is the region that can *not* be seen.

[4] This same point was made recently in Ref. [24], with reference to the choice of the "observer" who is defining the heat flux.

Note that in the approach of Ref. [1], with the new choice of Killing vector, one must still impose the condition that the expansion θ vanishes at p, because that condition is used in applying the Raychaudhuri equation to obtain the area change. By contrast, in the Noetheresque approach of the present paper the expansion at p will turn out to play no role whatsoever.

VI. PROPERTIES OF THE LOCAL HORIZON AND KILLING VECTOR

In this section we spell out the detailed geometric construction and properties of the local causal horizon (LCH) and Killing vector.

A. Local causal horizon

We define a local causal horizon H as follows. Consider any spacetime point p in a D-dimensional spacetime, and let Σ_p be any small patch of spacelike $(D-2)$-surface through p. The boundary of the past of Σ_p in the neighborhood of p has two components, each of which is a null surface generated by a congruence of null geodesics orthogonal to Σ_p. The LCH H is defined as one of these components.

As mentioned in the previous section, it will not matter for the present approach whether the congruence is stationary at p. However, if p is to be a stationary point, the expansion and shear of this congruence must vanish there. That is, the extrinsic curvature of Σ_p must vanish at p, which is equivalent to saying that Σ_p is generated by geodesics at p. For concreteness we will go further and assume that Σ_p is fully generated by geodesics emanating from p. This will be convenient for studying concrete examples of the approximate Killing vector and of entropy values.

B. Null normal coordinates

To establish the existence of an approximate Killing vector with the required properties, we will employ a "null normal coordinate" (NNC) system [25] adapted to the LCH. This is an explicit realisation of the coordinate systems first introduced in [3, 26] and is defined as follows. At a spacetime point p an orthonormal set of $(D-2)$ spacelike vectors $\{e^a_A\}$, $A = 1, 2, .., (D-2)$, is chosen, and a $(D-2)$-surface Σ_p is generated by geodesics with tangent vectors at p in the space spanned by $\{e^a_A\}$. The point reached on such a geodesic at unit affine parameter is assigned the coordinates x^A, when $x^A e^a_A$ is the tangent vector in that parametrisation at p. This defines standard Riemann normal coordinates on Σ_p based at p. A pair of future pointing null vector fields (k^a, l^a) normal to Σ_p is chosen on Σ_p, normalized such that $k_a l^a = -1$. Each point r in a small enough spacetime neighborhood of p lies on a unique geodesic orthogonal to Σ_p at some point q. Let the tangent to that geodesic at q be given by $V k^a + U l^a$ when r lies at unit affine parameter from q. The NNCs of r are then defined by[5]

$$x^\alpha_r = (U, V, x^A_q). \tag{14}$$

These coordinates are defined uniquely up to a rotation of the $(D-2)$-frame at p and a q-dependent rescaling of k^a, with inverse rescaling of l^a. The horizon H is the surface $U = 0$, restricted to $V \leq 0$. In particular, the central horizon generator Γ is the coordinate curve $U = 0$, $x^A = 0$, $V \leq 0$, with bifurcation point $p_0 \in \Gamma$ at $V = V_0 < 0$. Note that on H the coordinate V is an affine parameter along the null generators. The details of the construction, including the derivation of metric coefficients and Christoffel symbols, are presented in Ref. [25]. It is also shown there that the ambiguity in the choice of k^a can be exploited to make the coordinates locally inertial at p and to further specialize the properties of the metric components. A preferred choice is made by starting with the D-dimensional Riemann normal coordinates at p and adjusting them. This also induces a particular choice of affine parameter V.

C. Horizon slices

The Clausius relation applied to a horizon refers to the entropy change δS between two times. Those times are spacelike hypersurfaces, one to the future of the other, which intersect the horizon in two slices. For a black hole with a compact horizon, the two slices may bound a cylindrical region of the horizon.

For a local causal horizon, the considered process must be local, since the LCH is not even well defined except in a small neighborhood of the terminal point p. To localize the process we can restrict attention to cases where, rather than pushing the spacelike hypersurface forward in time everywhere, it is deformed to the future only in a small neighborhood of the bifurcation point p_0.[6] Then the two corresponding horizon slices also coincide everywhere except in a small region. Parts of two such slices, Σ_0 and Σ, are depicted in Fig. 2. Σ_0 corresponds to the $V = V_0$ surface and Σ lies to the future. If we truncate the horizon slices outside the region where they differ, their union $\Sigma_0 \cup \Sigma$ forms the closed boundary of a patch of the horizon. This will allow the difference of the entropies on the two slices to be computed using Stokes' theorem.

[5] We use greek indices, and upper case latin indices, for coordinates and coordinate components in the NNC coordinate system. Lower case latin indices are reserved for abstract indices on tensors.

[6] Hypersurface deformations of this sort were considered in Ref. [27] in the context of proving a local form of the Generalized Second Law.

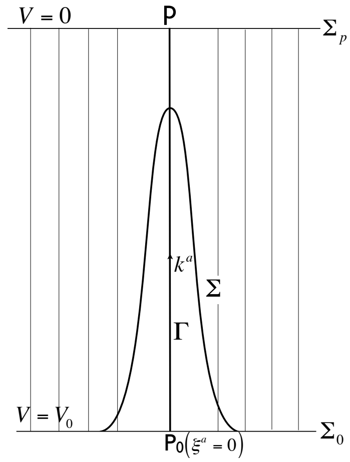

Figure 2: The local causal horizon is part of the boundary of the past of Σ_p, a spacelike $D-2$ dimensional surface. The vertical lines are horizon generators with affine parameter V and tangent vector k^a, and Γ is the central generator. Σ_p is geodesic at p, so the expansion θ and shear vanish at p. The horizontal lines represent constant V slices. The local Killing vector vanishes at the bifurcation point p_0. The entropy is compared on two slices, Σ_0 and Σ, which differ only in a compact region.

In fact, we shall need to further restrict the choice of horizon cuts so that they bound a *narrow* region of the horizon, because only then can the Killing identity be satisfied to sufficient accuracy for the approximations made in the derivation of the equation of state to be valid[7]. By "narrow", we mean that the ratio of the width in x^A to the length in V goes to zero in the limit as p_0 approaches p.

D. Local Killing vector

Next we define precisely the approximate Killing vector ξ^a that plays a central role in defining both the heat flux and, in the Noetheresque approach of Sec. VIII B, the entropy density. We will refer to this vector field as the "local Killing vector".

Of course a general curved spacetime has no Killing vectors. Nevertheless, in a small enough neighborhood of any point, any spacetime is approximately flat. In particular, in local inertial coordinates at p, the metric components take the form $g_{\alpha\beta} = \eta_{\alpha\beta} + O(x^2/L^2)$, where $\eta_{\alpha\beta}$ is the Minkowski metric, x denotes the coordinates, and L characterizes the shortest radius of curvature of the spacetime at p. (Here and below we use greek indices to refer to components in a particular coordinate system, reserving lower case latin indices for abstract tensor indices.) An approximate boost generator ξ^a can be defined in terms of local inertial coordinates by the formula for an exact flat spacetime boost generator, e.g. $(x, t, 0, 0)$ in Minkowski coordinates. This vector satisfies the Killing equation

$$\nabla_a \xi_b + \nabla_b \xi_a = 0 \tag{15}$$

exactly at p, and to $O(x)$ near p, but in general $\nabla_{(a}\xi_{b)}$ will have $O(x^2)$ terms. This is adequate for our purposes.

However, our application of the Clausius relation is also sensitive to the $O(x)$ part of the Killing identity

$$\nabla_a \nabla_b \xi_c = R^d{}_{abc}\xi_d. \tag{16}$$

For a true Killing field, this identity follows from the Killing equation (15). Conversely, the Killing identity implies the Killing equation if the latter holds at one point, because of the antisymmetry of the Riemann tensor in the last index pair. Our computations will rely on this identity being satisfied at $O(x)$, but for the approximate Killing field defined above it will generally not hold at that order. We can try to modify the definition of this vector field so as to satisfy both the Killing equation and the Killing identity at $O(x)$, but in fact this is not possible. However, narrowing our sights, ξ^b can be chosen so that the Killing identity holds to this order (in fact it can be chosen to hold exactly) on the single horizon generator Γ that ends on the terminal point p, and this turns out to be good enough. In effect, the local Killing symmetry can be extended away from p to a better approximation along a single null generator Γ than across the whole LCH. This calls to mind null Fermi coordinates[28], but we have used the NNCs to describe the situation since they are better adapted to the LCH.

Given that we confine the horizon to a narrow region surrounding the central horizon generator Γ, the integrals appearing in the Clausius relation will be dominated by their integrands evaluated on Γ. As such, any conditions that ξ^a may need to satisfy in order for the Clausius relation to lead to a consistent equation of state will be conditions imposed on Γ.

As motivated in Sec. V, the local Killing vector ξ^a is taken to have a bifurcation surface Σ_0—or at least a bifurcation point—at p_0 to the past of p, where it vanishes and where its covariant derivative generates boosts in the plane orthogonal to the bifurcation surface. The NNCs of p_0 are $(U, V, x^A) = (0, V_0, 0)$. It will be convenient to shift the affine parameter such that its origin coincides with the bifurcation point, i.e. we define $\tilde{V} \equiv V - V_0$, so p_0 lies at $\tilde{V} = 0$. We also want ξ^a to be approximately tangent to the generator Γ that connects p_0 to p, so that the evolution of the LCH can be interpreted as a small perturbation of a stationary background, which justifies

[7] An alternative option would be to restrict to a horizon patch that is symmetric in the transverse coordinates, to cancel contributions from terms linear in those coordinates. However, we consider this symmetry restriction to be artificial.

the use of the Clausius relation.[8] More specifically, the equation of state derivation will require that, at least to lowest order, ξ^a coincides with $\tilde{V}k^a$, as would be the case on a Killing horizon. And, finally, the derivation will require the Killing equation and the Killing identity to hold at $O(\tilde{V})$. Expressed in NNC components, the full set of requirements that must be imposed on ξ^α is:

$$\begin{aligned} \nabla_\alpha \xi_\beta|_{p_0} &= (k_\alpha l_\beta - l_\alpha k_\beta)|_{p_0}\,, &(17)\\ \xi^\alpha|_\Gamma &= \tilde{V}\delta^\alpha_V + O(\tilde{V}^2), &(18)\\ \nabla_{(\alpha}\xi_{\beta)}|_\Gamma &= O(\tilde{V}^2), &(19)\\ \nabla_\alpha\nabla_\beta\xi_\gamma|_\Gamma &= \left(R^\delta{}_{\alpha\beta\gamma}\xi_\delta\right)|_\Gamma + O(\tilde{V}^2). &(20)\end{aligned}$$

As alluded to above, ξ^a may be subjected to rather stronger conditions under which it approximates a true Killing vector more closely. In particular, we may choose ξ^a such that the Killing identity (16) holds *exactly* on Γ. This can be demonstrated by a perturbative argument which is given at the end of this section (see also [25] for further details). The identity then implies that the Killing equation also is exact on Γ, provided it holds exactly at one point. Another consequence is that ξ^a becomes exactly tangential to Γ, as will be shown next. Hence our claim is that all terms of quadratic or higher order in $\tilde{V}$ in (18) - (20) may actually be set to zero.

If a vector field ζ^a satisfies the Killing identity along a geodesic with affine tangent vector v^a, it follows that

$$v^a\nabla_a\left(v^b\nabla_b\zeta_c\right) = R_{dabc}\zeta^d v^a v^b. \qquad (21)$$

Eqn.(21) is an ordinary differential equation of second order and has a unique solution along the geodesic, once ζ^a and $v^b\nabla_b\zeta^a$ are given at an initial point. It is easily verified that $s\,v^a$ is a solution to (21), where s is any affine parameter on the geodesic, and $v^b\nabla_b(sv^a) = \dot{s}v^a$. Hence if $\zeta^a = sv^a$ and $v^b\nabla_b\zeta^a = v^a$ at one point, it follows that $\zeta^a = sv^a$ everywhere on the geodesic. Now recall that k^a is an affine tangent to Γ, and the local Killing vector is chosen to vanish at p_0 and to satisfy $\nabla_a\xi_b|_{p_0} = (k_a l_b - l_a k_b)|_{p_0}$. Taking $v^a = k^a$ thus yields $v^b\nabla_b\xi^a\big|_{p_0} = v^a|_{p_0}$, so we may conclude that $\xi^a = \tilde{V}k^a$ everywhere on Γ.[9]

As regards the Killing equation, we recall that the NNC system is local inertial at p. As discussed at the beginning of the current section, it is then possible for the Killing equation (15) to hold at $O(x)$, and this turns out to be consistent with the Killing identity on Γ. In NNCs, the approximate Killing equation implies that the components of the local Killing vector are of the form $\xi^\alpha = \tilde{V}\delta^\alpha_V - U\delta^\alpha_U + O(x^3)$.

In summary, the main properties exhibited by our choice of local Killing vector are given by (17) together with

$$\begin{aligned} \xi^\alpha|_\Gamma &= \tilde{V}\delta^\alpha_V, &(22)\\ \nabla_{(\alpha}\xi_{\beta)} &= O(x^2), &(23)\\ \nabla_\alpha\nabla_\beta\xi_\gamma|_\Gamma &= \left(R^\delta{}_{\alpha\beta\gamma}\xi_\delta\right)|_\Gamma\,. &(24)\end{aligned}$$

To conclude this section, we turn to the perturbative argument that the Killing identity (16) can be satisfied exactly on Γ. We specify ξ^a in a neighborhood of Γ by its (covariant) components in NNCs as the Taylor series

$$\xi_\alpha = U\delta^V_\alpha - \tilde{V}\delta^U_\alpha + C_{\beta\gamma\alpha}\tilde{x}^\beta\tilde{x}^\gamma + D_{\beta\gamma\delta\alpha}\tilde{x}^\beta\tilde{x}^\gamma\tilde{x}^\delta + \ldots (25)$$

where $\tilde{x}^\alpha = x^\alpha - V_0\delta^\alpha_V$ and in particular $\tilde{V} = V - V_0$. Similarly, on Γ the deviation away from the Killing identity is written as a power series in the affine parameter $\tilde{V}$. As shown in [25], the latter series may be set to zero order by order through an appropriate choice of the expansion coefficients occurring in (25).

More specifically, at the linear order (20) needed for the equation of state derivation, one finds

$$\begin{aligned} C_{\alpha\beta\gamma} &\sim V_0 \times (\text{Riemann components at } p), &(26)\\ D_{V\alpha\beta\gamma} &\sim (\text{Riemann components at } p). &(27)\end{aligned}$$

Thus the terms that are quadratic and cubic in $\tilde{x}$ in the expansion (25) both contribute to ξ_α only at $O(x^3)$, counting V_0 as $O(x)$. This is consistent with the Killing equation (23) at $O(x)$, from which one finds that quadratic order terms must be absent in (25).

For the purpose of the general derivation of field equations all we need to know is that the conditions (17)-(20) can be met. For the purpose of computing the actual entropy (which will have a dependence on the local Killing vector, see Sec. VIII B) and comparing with the area in the GR case or with other expressions in higher derivative gravity, the detailed form of the local Killing vector is generally needed. For more explicit expressions of the higher order coefficients in (25) we refer to [25].

VII. BLACK HOLE ENTROPY AS NOETHER CHARGE

In this section we review Wald's expression for black hole entropy in terms of the Noether charge[2, 3]. This expression will motivate the form of LCH entropy that we adopt.

Consider a diffeomorphism-invariant Lagrangian field theory of gravity in arbitrary dimensions. A variation of

[8] Moreover, if we were to let the local Killing vector and the tangent to the generator vary independently, it appears the Clausius relation would be overly restrictive.

[9] It will generally be inconsistent with the Killing identity on Γ to choose ξ^a to be tangential to *all* the generators of a small horizon patch. Moreover, if the horizon expansion does not vanish at the bifurcation point, such a choice would also be inconsistent with the Killing equation imposed at linear order in a spacetime neighborhood.

the action is always equal to a sum of terms proportional to the field equations and a surface term. When the variation is induced by a vector field ξ^a, this equality can be expressed as the statement that a certain current j^a is closed, i.e. $\nabla_a j^a = 0$. The current is closed even when evaluated off-shell and takes the form

$$j^a = \theta^a - L\,\xi^a - \left(2\,E^{ab} - T^{ab}\right)\xi_b + ..., \tag{28}$$

where the symplectic current θ^a stems from the surface term, L is the Lagrangian scalar in the action, $2\,E^{ab} - T^{ab} = 0$ is the metric field equation and the dots indicate off-shell terms corresponding to the matter field equations. The Noether current J^a is defined as $J^a = \theta^a - L\xi^a$, and on shell coincides with the closed current j^a. Furthermore, since the on-shell Noether current is closed for any possible vector field ξ^a, it can be expressed as $J^a = 2\nabla_b Q^{ab}$, where the antisymmetric tensor Q^{ab} is a local function of the fields and their derivatives[29]. The tensor Q^{ab} is referred to as a Noether potential.[10]

Since the vector field ξ^a only enters the expression for a Noether potential when taking Lie derivatives of tensor fields with respect to ξ^a, any Noether potential is of the form

$$Q^{ab}[\xi] = W^{abc}\xi_c + P^{abcd}\nabla_c\xi_d, \tag{29}$$

apart from the possible addition of a total divergence. More specifically,

$$Q^{ab}[\xi] = W^{abc}\xi_c + X^{abcd}\nabla_{[c}\xi_{d]} + Y^{ab} + \nabla_c Z^{abc}, \tag{30}$$

where the tensors W, X, Y, Z are locally constructed from the dynamical fields, Y is linear in Lie derivatives with respect to ξ, and Z^{abc} is totally antisymmetric. The decomposition of Q in terms of W, X, Y, Z is not unique, and in addition the Noether potential has three sources of ambiguity, coming from the freedom to add a total divergence to the Lagrangian, the symplectic potential, or the Noether potential itself. Using all this freedom Y and Z can be set to zero, X can be chosen [3] as

$$X^{abcd} = -\frac{\partial L}{\partial R_{abcd}} + ... \tag{31}$$

and W is then given by [30]

$$W^{abc} = 2\nabla_d X^{abcd} + \text{matter terms} + ..., \tag{32}$$

where the dots indicate terms that stem from derivatives of the Riemann tensor in the Lagrangian.

If the theory admits stationary black hole solutions with a regular bifurcation surface, variations away from these solutions satisfy the first law (1), with the entropy S_{BH} defined by [3]

$$S_{BH} = \frac{2\pi}{\hbar}\oint_\Sigma Q^{ab}[\hat{\chi}]N_{ab}dA. \tag{33}$$

Here, Σ denotes any slice of the stationary horizon, $N_{ab} = 2k_{[a}l_{b]}$ is its binormal (normalized as $N_{ab}N^{ab} = -2$), dA is the $(D-2)$ dimensional area element, and $\hat{\chi}^a$ is the horizon-generating Killing vector normalized to unit surface gravity. The integral is referred as the Noether charge and is invariant under the three sources of ambiguity [3, 5]. Invoking the stationary symmetry, it was further shown in these references that the entropy may equally be expressed by substituting $Q^{ab}[\hat{\chi}]$ in (33) with $X^{abcd}N_{cd}$. Thus the W term does not contribute to the black hole entropy. For general relativity, S_{BH} becomes the familiar Bekenstein-Hawking entropy $A/4\hbar G$.

VIII. CLAUSIUS RELATION AND EQUATION OF STATE

In this section we adopt the assumption that LCH's have an entropy of Noetheresque form, i.e. the entropy density depends on the local Killing vector in the same way that the Noether potential (29) depends on the horizon generating Killing field. With this entropy, we shall find that the Clausius relation applied to all LCHs, together with the local conservation law for the matter stress tensor, can be satisfied provided that i) the entropy density can be identified with (the gravitational part of) a Noether potential of some Lagrangian, and ii) the fields satisfy the metric field equation for that Lagrangian.[11] This is the equation of state for the corresponding entropy function.

A. Clausius relation for a local causal horizon

This Clausius relation for a LCH is analogous to the first law for stationary black holes. However, the latter relies on the stationarity of the black hole horizon, as well as the fact that a horizon slice is a complete boundary component of a spacetime slice, neither of which hold for a LCH. Nevertheless, it turns out that the construction can be localized enough to make the Clausius relation at least well defined. First, as explained in Sec. VI C and Fig. 2, we compare the entropy on two LCH slices Σ and Σ_0 that share a common boundary, so that together they form the boundary of a local patch H of the horizon. The

[10] In this paper we use tensor notation. The formalism in Refs. [2, 3] makes use of differential forms, which are dual to these tensors. The relation between a p-form F and the corresponding tensor $\tilde{F}$ is $F_{a_1\ldots a_p} = \tilde{F}^{b_1\ldots b_{n-p}}\epsilon_{b_1\ldots b_{n-p}a_1\ldots a_p}$, where ϵ is the volume element. The tensor corresponding to the exterior derivative dF is $(n-p)\nabla_a\tilde{F}^{b_1\ldots b_{n-p-1}a}$.

[11] Note that for the equation of state to be well defined the matter stress tensor need not be the functional derivative with respect to the metric of an action.

lack of stationarity looks more problematic because, for a dynamical black hole horizon in a general theory, there is no well defined notion of entropy available[12]. The problem is that there is no Killing vector and, if a vector field is somehow selected, the ambiguities in the definition of the Noether charge will make the entropy ambiguous, unlike for a stationary black hole horizon.[13] On the other hand, locally there is always an approximate Killing vector, the "local Killing vector" constructed in Sec. VI D. This turns out to act enough like a Killing vector to make the Clausius relation well defined in a certain local limit, once a form for the entropy is adopted.

In the context of horizon thermodynamics it is natural to expect that the entropy is an extensive quantity. Thus we assume that it can be expressed as an integral of a $(D-2)$-form over a spacelike slice Σ of the horizon. We adopt the dual description, and express the entropy as the integral

$$S = \int_\Sigma s^{ab} N_{ab} dA, \tag{34}$$

where the entropy density s^{ab} is an antisymmetric tensor, N^{ab} is the binormal to the slice and dA is the area element on the slice. The change of this entropy between the two horizon slices is

$$\begin{aligned} \delta S &= S - S_0 \\ &= \oint_{\Sigma\cup\Sigma_0} s^{ab} N_{ab} dA \\ &= 2\int_H \nabla_b s^{ab} dH_a \\ &= -2\int_H \nabla_b s^{ab}\, k_a dV dA, \end{aligned} \tag{35}$$

where Σ_0 is taken with the opposite orientation to Σ, and in the third line we have used Stokes' theorem (for details of Stokes' theorem on a null surface see Ref. [31]).

On the right hand side of the Clausius relation, $dS = \delta Q/T$, we have the heat flux

$$\delta Q = \int_H (-T_{ab})\xi^b dH_a \tag{36}$$

divided by the Unruh boost temperature $T = \hbar/2\pi$. The heat flux integrand is proportional to the local Killing vector which vanishes at p_0 and is thus of $O(x)$ in NNCs in the neighborhood of p_0. The Clausius relation is imposed in the limit $p_0 \to p$, which means that the entropy change integrand of (35), multiplied by T, must be equal to the $O(x)$ heat flux integrand, up to $O(x^2)$ terms. That is,

$$-(\hbar/\pi)\nabla_b s^{ab}\, k_a = T^{ab}\xi_b\, k_a + O(x^2). \tag{37}$$

This relation is imposed at all points p and for all null directions at p.

We emphasize that the limit $p_0 \to p$ is taken in a formal sense only, in order to identify the leading order contribution for a small region. From a physical point of view, it makes no sense to consider an arbitrarily small region, because quantum fluctuations of the metric presumably invalidate our semiclassical considerations at sufficiently short distances.

B. Entropy density of Noetheresque form

Inspired by Refs. [16, 19], we propose a local entropy density of the same form as the Noether potential of Eqn. (29), where ξ^a is the local Killing vector, but a priori no restrictions are placed on W and P, other than that they are constructed locally from the dynamical fields. We emphasize that, although we are using a similar notation, we do *not* assume at this stage that the divergence of the local entropy density is a Noether current associated to a Lagrangian, in contrast to the entropy density of stationary black holes. This form of entropy density can be viewed as the most general one that depends linearly on a local Killing vector that satisfies the Killing identity, and contributes to the entropy change at $O(x^2)$. As mentioned in the introduction, since the local Killing vector determines the local notion of equilibrium and heat, it is not entirely unnatural that the entropy would depend on the Killing vector. On the other hand, this entropy is not strictly intrinsic to the horizon, and has no immediate statistical interpretation that we are aware of.

We thus assume the entropy density takes the form

$$s^{ab} = \frac{2\pi}{\hbar} Q^{ab}, \tag{38}$$

with

$$\begin{aligned} Q^{ab} &= W^{abc}\xi_c + P^{abcd}\nabla_c\xi_d \\ &= W^{abc}\xi_c + \left(X^{abcd} + Y^{abcd}\right)\nabla_c\xi_d. \end{aligned} \tag{39}$$

In the second line of (39), the tensor P is split into an antisymmetric part X and a symmetric part Y:

$$X^{abcd} = X^{ab[cd]} \quad \text{and} \quad Y^{abcd} = Y^{ab(cd)}. \tag{40}$$

We could of course absorb the factor of $2\pi/\hbar$ into the definition of Q^{ab}. The only reason we factor it out here is so that the notation will make the analogy with the first law for stationary black hole horizons more transparent.

We now show that the symmetric part does not contribute to the Clausius relation. The contribution of

[12] General relativity is an exception, where the area of the horizon slices is a good candidate for the dynamical entropy. This is motivated by the area theorem, as well as by the form of entanglement entropy; see Sec. II.

[13] Although the proposed dynamical entropy of Ref. [3] (in terms of the instantaneous boost invariant metric) manages to bypass the ambiguities, it most likely does not satisfy a second law[32].

Y^{abcd} to $\nabla_b Q^{ab}$ is

$$\nabla_b \left(Y^{abcd}\nabla_{(c}\xi_{d)}\right) = \nabla_b Y^{abcd}\,\nabla_{(c}\xi_{d)} + Y^{abcd}\nabla_b\nabla_c\xi_d. \tag{41}$$

Using the approximate local Killing equation (23), we see that the first term on the RHS is of $O(x^2)$. The second term is symmetric in cd, so without any change we may add a term in the Riemann tensor, yielding

$$Y^{abcd}\left(\nabla_b\nabla_c\xi_d - R^f{}_{bcd}\,\xi_f\right), \tag{42}$$

which according to the approximate Killing identity (20) is of $O(x^A)$, where x^A are the transverse coordinates of the NNC system. The contribution of (41) is therefore negligible in the limit of small, narrow horizon patches.

Thus, at least for the part that contributes to the entropy change, our proposal for the entropy density reduces to $2\pi/\hbar$ times

$$Q^{ab} = W^{abc}\xi_c + X^{abcd}\nabla_c\xi_d, \tag{43}$$

where W and X are, so far, unspecified tensors, constructed locally from the dynamical fields, antisymmetric in the first two indices, and in addition X is antisymmetric in the last two indices. Note that the X term is of $O(1)$, and the W term is of $O(x)$. Had we kept the Y term it would have been of $O(x^2)$.

Next we show that the Clausius relation requires that W is a combination of divergences of X. The divergence of (43) can be written as

$$\begin{aligned}\nabla_b Q^{ab} &= \left(\nabla_r W^{arb} + X^{arst}R^b{}_{rst}\right)\xi_b \\ &+ X^{arst}\left(\nabla_r\nabla_s\xi_t - R^b{}_{rst}\,\xi_b\right) \\ &+ \left(W^{ast} + \nabla_r X^{arst}\right)\nabla_s\xi_t.\end{aligned} \tag{44}$$

Again, the first term is proportional to the local Killing vector and hence of $O(x)$, while the second term involving the Killing identity may be neglected on narrow patches. It is precisely here that the need to impose the Killing identity arises. Were this identity not satisfied by ξ^a, the second term would make a contribution at the same order as the first term, which would depend on the higher order coefficients in the local Killing vector expansion. This would ruin our extraction of the local field equation from the Clausius relation and would probably make it impossible to consistently impose the Clausius relation at all for such local Killing vectors.

In the third term, the symmetric part in (st) is of $O(x^2)$ by virtue of the Killing equation, but a priori the antisymmetric part is of $O(x^0)$. On the other hand, the integrand in the heat flux is of $O(x)$, so in order to be consistent with the Clausius relation (37), the $O(x^0)$ part of the third term must vanish. It follows that the local Clausius relation can only be satisfied at all spacetime points if W and X are related via

$$W^{a[st]} + \nabla_r X^{arst} = 0. \tag{45}$$

Because W is antisymmetric in its first two indices, this relation completely determines W as a function of X, to wit

$$W^{arb} = \nabla_s\left(X^{sarb} + X^{sbra} + X^{srba}\right). \tag{46}$$

Thus only the first term in (44) survives.

Notice that although the W term does not contribute to the entropy at p_0, nor at p in the limit $p_0 \to p$, its rate of change is comparable to that of the X term, so that—in contrast to the stationary comparison version of the first law of black hole mechanics [3]—it makes an important contribution to the Clausius relation, which could not be satisfied without it.[14]

C. Equation of state

Substituting (46) into the first term of (44), and using the fact that $\xi_a \propto k_a$ on the central generator, the validity of the Clausius relation (37) for all k^a implies

$$R^{(a}{}_{rst}X^{b)rst} + 2\nabla_r\nabla_s X^{(a|s|b)r} + \Phi g^{ab} = -\frac{1}{2}T^{ab}, \tag{47}$$

where Φ is some scalar function that may depend on the metric and curvature. Note that (47) follows from the Clausius relation irrespective of the values of expansion and shear anywhere on the horizon patch.

As was done in Sec. IV, we impose local conservation of energy-momentum to determine the function Φ. Then Eqn. (47) leads to

$$\nabla^a\Phi = -\nabla_b\left(R^{(a}{}_{rst}X^{b)rst} + 2\nabla_r\nabla_s X^{(a|s|b)r}\right). \tag{48}$$

In order for such a Φ to exist, the right hand side must be the gradient of a scalar. This integrability condition further constrains the nature of X, which so far is only required to be anti-symmetric in both the first and second pair of indices.

If the integrability condition is satisfied, then the left hand side of (47) is a divergence free tensor constructed from the metric and its derivatives. One way to obtain such a tensor is from the variational derivative of a scalar action functional with respect to the metric, $\delta I_g[g]/\delta g_{ab}$,

---

[14] One might think that perhaps a total divergence can be added to the entropy density such that the W term is cancelled. That is, if a tensor $Z^{abcd} = Z^{[abc]d}$ could be found such that $W^{abd} = \nabla_c Z^{abcd}$, then we would have $Q^{ab} = (X^{abcd} - Z^{abcd})\nabla_c\xi_d + \nabla_c(Z^{abcd}\xi_d)$. The total divergence term would not affect the changes in the entropy from one slice to another with common boundary, so for the Clausius relation this would be equivalent to having $W = 0$ and replacing X by $X - Z$. The existence of such a Z is at first glance conceivable since, according to (46), W must be a divergence. However, the required total antisymmetry of Z is not shared by the combination of X's in (46). Only the divergences of Z and X enter the equation, but still it seems unlikely that such a Z exists in general.

which is automatically divergence free. In fact, it was argued in Ref. [33] that *all* such tensors arise in this way. If so, then the 'Clausius equation' (47) is precisely the equation of motion that derives from the action $I_g + I_{\rm matter}$, with an undetermined cosmological constant.

1. Relation between entropy density and Lagrangian

We can be more specific about the relation between the entropy and an action whose equation of motion is (47). If a gravitational Lagrangian $L[g_{ab}, R^a{}_{bcd}]$ is a scalar formed algebraically from the metric and Riemann tensor, then the corresponding equation of motion is precisely (47), [15] with

$$X^{abcd} = -\frac{\partial L}{\partial R_{abcd}}, \tag{49}$$

and $\Phi = L/2$. Note that if X is assumed to have this form, it has all the symmetries of the Riemann tensor. Using these symmetries one can show that our condition (46) that determines W in terms of X becomes exactly the same as Eqn. (32). Thus, with the sufficient condition (49) assumed, our conjectured entropy density is nothing but the specific choice discussed in Sec. VII for the Noether potential associated to a Lagrangian $L[g_{ab}, R^a{}_{bcd}]$. [16]

For a Noetheresque entropy density with X of the form (49), and W satisfying (45), the divergence in Eq. (44) is nothing but the closed current (28) of Sec. VII, for the Lagrangian scalar $L[g_{ab}, R^a{}_{bcd}]$. That is,

$$2\nabla_b Q^{ab} = \theta^a - L\xi^a - 2E^{ab}\xi_b, \tag{50}$$

where E^{ab} is the variational derivative of L, given by the LHS of eqn (47). Specifically, the first term of the RHS of (44) is given by $-L\xi^a - 2E^{ab}\xi_b$, while the symplectic current θ^a is given by the last two terms. We further note that the symplectic current vanishes for a Killing vector, while it was found here to be negligible for the local Killing vector. Similar remarks were also made in [19].

Finally, we point out that this particular entropy density does not include a dependence on matter fields. However, such a dependence should perhaps not be ruled out a priori.

[15] To show this is a bit tricky. One can regard the Lagrangian as a function of g_{ab} and R_{abcd}, and use the identity $\partial L/\partial g_{ab} = -2R^{(a}{}_{rst}\delta^{b)}_q \partial L/\partial R_{qrst}$. This identity can be easily established by exploiting the fact that the Lagrangian can also be considered as a function of the $(2, 2)$ tensor $R^{ab}{}_{st}$, without any explicit dependence on the metric or R_{qrst}, and taking the partial derivatives using the chain rule and $R^{ab}{}_{st} = g^{aq}g^{br}R_{qrst}$. For a detailed derivation, see Ref. [34].

[16] We note that not all Noether potentials associated to a given Lagrangian satisfy the condition (46), it is curious that the local Clausius relation restricts the form of the Noether potential in this manner. Perhaps this stems from the lack of exact symmetry in a general spacetime background.

D. Examples

1. $L(R)$ theories

We now illustrate our result by a simple example, with X given by

$$X^{abcd} = \frac{f}{2}\left(g^{ac}g^{bd} - g^{ad}g^{bc}\right), \tag{51}$$

where $f = f(R, R_{ij}, R_{ijkl})$ is an arbitrary scalar function. This expression satisfies all the symmetry requirements imposed so far, but from Eq. (48) we obtain

$$\nabla^a\Phi = -\frac{1}{2}f\nabla^a R, \tag{52}$$

which can not be integrated unless the function f depends only on the Ricci scalar. Thus, we find a strong restriction on the form of the entropy density if the Clausius relation is to be consistent with energy conservation. For example, the leading order term in the entropy density can not be, say, $1 + \alpha R_{ab}R^{ab}$.

If f is only a function of R, it can be written as $f = -dL/dR$ for some function $L(R)$, and then we have $\Phi = L/2$ (the arbitrary additive constant freedom in Φ can be absorbed into L, and corresponds to a cosmological constant). With f of this form, Eq. (47) becomes

$$\begin{aligned} L'(R)R_{ab} \;&-\; \nabla_a\nabla_b L'(R) \\ &+\; \left(\Box L'(R) - \frac{1}{2}L\right)g_{ab} = \frac{1}{2}T_{ab}. \end{aligned} \tag{53}$$

This is identical to the equation of motion that results from the Lagrangian $L(R) + L_{\rm matter}$. In the case of GR, $L = R/(16\pi G)$, and we obtain the usual Einstein equation.

In the case of generic $L(R)$ theories, we have thus obtained the field equation as an equation of state from the Clausius relation, without the need for a term representing internal entropy production due to bulk viscosity, unlike in the non-equilibrium framework of Ref. [12]. As described in Sec. IV, in that approach the change of the entropy density $f(R)$ coming from the gradient of the background R at the equilibrium point is cancelled by a choice of non-zero horizon expansion, which then leads to the bulk viscosity term in the entropy balance equation. Instead, with the Noether charge entropy density, the W term is chosen so that the Clausius relation can be satisfied without internal entropy production.

2. Lovelock theories

Suppose that X^{abcd} has no higher than second derivatives, and that also $\nabla_a X^{abcd} = 0$, so (46) implies $W^{abc} = 0$. Then it follows that the tensor on the left hand side of the field equation (47) is second order in derivatives. Moreover, after imposing the integrability condition (48),

we infer that this tensor, built from the metric and its first and second derivatives, must be identically divergence free. The only such tensors come from the metric variation of a Lovelock Lagrangian[35], so the assumed properties of such an X^{abcd}, together with the Clausius relation, imply that X^{abcd} arises as in (49) from a Lovelock Lagrangian of the form $L = L[g_{ab}, R^a{}_{bcd}]$.

3. *General relativity*

In example (51) with the choice $f = -(16\pi G)^{-1}$, Eqn. (46) implies that $W = 0$. The corresponding entropy density obtains from Eqns. (43) and (38) and reads $s^{ab}_{GR} = -(8\hbar G)^{-1}\nabla^{[a}\xi^{b]}$. The steps of Sec. VIII C then lead to the Einstein equation of state.

For the Killing vector (25) we have $\nabla^{[a}\xi^{b]} = 2k^{[a}l^{b]} + O(x^2)$. Any horizon slice has a local binormal of the form $N_{ab} = 2k_{[a}l_{b]} + k_{[a}m_{b]}$ where m^b is some spacelike vector tangent to the horizon and therefore orthogonal to k^a. For generic slices, the resulting entropy (34) coincides to leading order with the area in units of $1/4\hbar G$.

The *difference* in entropy between two slices requires a more careful analysis. On a slice of constant affine parameter V, the entropy will differ from the area at subleading order, as it does on any slice. The coefficients of the terms of subleading order are given in terms of the coefficients C and D of Eqn. (25), many of which are in turn set by the Killing identity (20) imposed on Γ. These coefficients can be shown to be such that the subleading order terms, in the case of a slice of constant V, involve only transverse coordinates. Therefore, when comparing two slices of constant affine parameter, the subleading terms drop out and the entropy difference to leading order coincides with the area difference.

However, for the Clausius relation of Sec. VIII A, we need to compare two slices with common boundary. In that case, on at least one of these slices the mismatch between entropy and area at subleading order does depend on V, in such a way that the entropy and area differences will not generically coincide to leading order, unlike in the first law of global horizon mechanics in GR or the original Einstein equation of state derivation [1]. The easiest way to see this is to employ Stokes' theorem (35) with the entropy density $s^{ab}_{GR} = -(8\hbar G)^{-1}\nabla^{[a}\xi^{b]}$. To evaluate the integrand of (35), we use (44) and recall that the approximate Killing vector was chosen such that (44) is dominated by its first term. In this way we can express the entropy difference as

$$\delta S \approx \int_H R_{ab}\, k^a k^b\, (V - V_0)\, dV dA, \qquad (54)$$

where the integration range H is a patch of the LCH enclosed by two slices. This integral will be equal to the area difference to leading order only for a very special choice of slices, as we now describe.

Suppose that the first slice Σ_0 is the surface $V = V_0$, and the shape of the second slice Σ is such that the en-

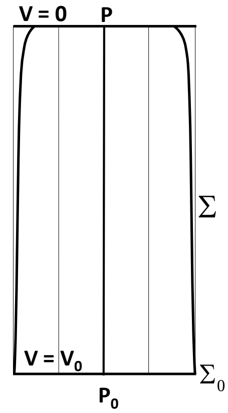

Figure 3: A local causal horizon patch enclosed by a special pair of slices. The entropy change associated with $s^{ab}_{GR} = -(8\hbar G)^{-1}\nabla^{[a}\xi^{b]}$ is equal at leading order to the area change over $4\hbar G$ if (*i*) the horizon expansion and shear vanish at p, and (*ii*) the patch between the slices Σ_0 and Σ nearly coincides with the interval $[V_0, 0]$ across the entire patch.

closed horizon patch nearly amounts to the entire interval $[V_0, 0]$ (see Fig. 3). Then, as indicated in (13), the entropy difference may equivalently be approximated by $\delta S \approx \int_H R_{ab}k^a k^b\, V dV dA$. If in addition the expansion and shear of the horizon generators vanish at p, then the Raychaudhuri equation tells us that $\delta S \approx \int_H \theta\, dV dA$. Since the integration range is nearly the entire interval $[V_0, 0]$ this evaluates to $\delta S \approx A(V = 0) - A(V = V_0)$, where $A(V)$ denotes the area of a slice of constant affine parameter V. The area of Σ nearly coincides with $A(V = 0)$ because its steep sides are "nearly null" and hence contribute negligibly to the area. Clearly, if the two slices are not chosen in this special fashion the entropy difference will not coincide to leading order with the area difference.

Nevertheless, if the entropy difference between two generic slices with common boundary obeys the local Clausius relation, the Einstein equation must follow as a consequence. We can illustrate the way this works with a trivial example in flat spacetime. A small patch of a light cone may be considered to be part of a LCH, enclosed by two spacelike slices with common boundary. The boost energy flux is identically zero, so the Clausius relation tells us the entropy on both slices must be identical. Employing s^{ab}_{GR}, this may be easily verified. However, the area of these slices is clearly not identical.

The entropy density s^{ab}_{GR} also shows it is not only thermodynamically more natural to place the bifurcation point of the local Killing vector field to the past of the terminal point (rather than coincident with the terminal point) as was discussed in Sec. V, it is actually required if we want the contribution to the entropy on a constant V slice in the GR case to correspond to the area rather than minus the area.

To see why this is so, consider the fact that in the

local Noether charge approach we have derived the Einstein equation as an equation of state, without specifying where the bifurcation point lies. However, if we want the entropy to agree with the area (to leading order), then $s^{ab}N_{ab}$ must be a positive constant. In the case of GR, the entropy density is $s_{ab} \propto -\nabla_{[a}\xi_{b]} \sim -\nabla_a \xi_b$, so $s^{ab}N_{ab} \sim -k^a l^b \nabla_a \xi_b$. Since to leading order the causal horizon is a Killing horizon, we have $k^a \nabla_a \xi_b \propto \pm\xi_b \propto \pm k_b$, where the $+$ sign holds when the bifurcation point lies to the past, so that the Killing vector is stretching in the future direction, and the $-$ sign holds when the bifurcation point lies to the future, so the Killing vector is shrinking (recall that we take the Killing vector to be future pointing on the horizon). Hence $s^{ab}N_{ab} \sim \mp l^b k_b = \pm 1$. To have a positive entropy we must therefore take the bifurcation point in the past. We have no option to reverse the sign of the assumed entropy density, since that would lead to a field equation with the opposite sign for the gravitational constant. That is, a negative entropy change would arise from a positive heat flux.

On the other hand, choosing the local Killing vector in this way entails a disturbing feature. If the proposed entropy represents the state of the horizon, one would expect it, and its difference, to depend only on quantities intrinsic to the horizon. However, the entropy difference is given by the boost energy flux, which depends on the local Killing vector, which in turn depends on a freely chosen location V_0 of the bifurcation surface Σ_0. This is unsatisfying, as V_0 is a freely chosen parameter and does not encode any intrinsic property of the LCH.

Finally, we point out that the entropy density s^{ab}_{GR} is only one among an infinite number to give rise to the Einstein equation since, by Stokes' theorem, addition to the entropy density of any total divergence will have no influence on the entropy difference This is in keeping with the fact that the Clausius relation is a thermodynamic relation, from which the value of the statistical entropy can only be deduced up to addition of an arbitrary constant.

E. Including derivatives of curvature

On general grounds, one may expect strong gravitational fields to be governed by a field equation derived from a Lagrangian that includes covariant derivatives of the Riemann tensor. In the present context, this raises the question whether such a field equation may arise as an equation of state corresponding to an entropy density of the form (43), introduced in Sec. VIII B. As was discussed in that section, in order for this to be possible the tensors W and X must be related by Eqn. (46). This may be possible, but we point out that the specific choice of Sec. VII for the Noether potential of such a Lagrangian does not satisfy the relation (46). The problem is that the divergence of this Noether potential contains a Lie derivative of the Riemann tensor with respect to the local Killing vector. Whereas such a derivative would vanish for a true Killing vector, it must be expected to be of order one in a general spacetime. Furthermore, it does not appear that relation (46) can be salvaged by appealing to the freedom to add total divergences at the different stages in the construction of the Noether potential. If this is true it would mean that the strict analogy to the first law of black hole mechanics breaks down, since that law applies for any diffeomorphism-invariant Lagrangian.

However, it would still leave open the possibility that entropy densities of the form (43) exist that do satisfy relation (46) and give rise to the field equations of high derivative Lagrangians, but that are not Noether potentials associated with such Lagrangians. As shown in the Appendix, the integrability condition that must be satisfied in order for such entropy densities to exist (whether or not they are Noether potentials) may be written in the form of a pair of tensor equations of first order, at least for Lagrangians that depend on no more than first order covariant derivatives of the Riemann tensor.

IX. CONCLUSION

The question we addressed is whether gravitational field equations with higher derivative terms can be derived from the Clausius relation applied to a higher derivative horizon entropy. First we discussed problems that arise with previous approaches to this problem. Then we adapted the starting point of one of those approaches, and assumed that horizon entropy depends on an approximate local Killing vector in a way that mimics the diffeomorphism Noether charge that yields the entropy of a stationary black hole. We showed that the problems can all be avoided by a careful choice of the nature of the horizon patch to which the Clausius relation is applied. In particular, the Clausius relation must refer to the change of entropy between two slices of the horizon that together form the complete boundary of a patch, and this patch must be narrow enough to neglect violations of the Killing identity. We exploited a power series expansion in a coordinate system adapted to the horizon to establish the required properties of the local Killing vector.

Together with matter energy conservation, the Clausius relation applied to all such local horizon patches leads to an integrability condition on the assumed horizon entropy density. We showed that this condition can be satisfied if the latter is in fact a Noether potential associated with a Lagrangian constructed algebraically from the metric and Riemann tensor. In that case the Clausius relation implies that the field equation for that Lagrangian holds. We have not proved that this is the only way to satisfy the integrability condition, but that may be the case. In particular, the field equation for a theory with derivatives of curvature in the Lagrangian is unlikely to be obtained in this way using for the entropy density a Noether potential derived from the Lagrangian,

although it might conceivably arise from a different entropy density.

The higher derivative extension of the equation of state derivation was achieved in this paper at the cost of introducing a dependence of the entropy on the choice of local Killing vector. Whereas this dependence occurs at subleading order, the entropy difference depends on the local Killing vector at leading order. It is therefore not clear to us whether the derivation has any thermodynamic significance. We regard it as a positive answer to a technical question, but its physical interpretation remains obscure. Perhaps the steps we have taken are a valid part of a picture that will only become clear once so far overlooked subtleties are taken into account. For example, the arbitrariness of an additive constant in thermodynamic entropy, the contribution from the entanglement entropy, and the related need to regularize by some kind of subtraction or comparison might play an important role in formulating the local thermodynamics of the vacuum. Then again, it may just be that the contribution of higher curvature terms to a gravitational field equation cannot be sensibly captured at a local thermodynamic level.

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Appendix A: Integrability condition for equations of state corresponding to Lagrangians $L[g_{ab}, R^a{}_{bcd}, \nabla_f R^a{}_{bcd}]$.

For notational convenience, we define the tensor Z [5] by

$$Z^{f:abcd} \equiv \frac{\partial L}{\partial\left(\nabla_f R_{abcd}\right)}. \tag{A1}$$

Then the choice of Noether potential of Sec. VII for a Lagrangian $L[g_{ab}, R^a{}_{bcd}, \nabla_f R^a{}_{bcd}]$ is given by

$$Q^{ab}_{(0)} = W^{abc}_{(0)}\xi_c + X^{abcd}_{(0)}\nabla_c\xi_d, \tag{A2}$$

where

$$X^{abcd}_{(0)} = -\frac{\partial L}{\partial R_{abcd}} + \nabla_f Z^{f:abcd}, \tag{A3}$$

$$W^{abc}_{(0)} = 2X^{abcd}_{(0)} + a:[bc] - b:[ac] - c:[ab] \tag{A4}$$

and

$$a:[bc] \equiv Z^{a:[b}{}_{def}R^{c]def}. \tag{A5}$$

Due to the terms $a:[bc]$ etc., the relation (46) is not satisfied for $Q^{ab}_{(0)}$. We now suppose that the entropy density consists of the Noether potential $Q^{ab}_{(0)}$ with a further Noetheresque potential added. That is, we take

$$s^{ab} = 2\pi/\hbar\, Q^{ab}, \tag{A6}$$

where $Q^{ab} = Q^{ab}_{(0)} + Q^{ab}_{(1)}$ and $Q^{ab}_{(1)} = W^{abc}_{(1)}\xi_c + X^{abcd}_{(1)}\nabla_c\xi_d$.

The divergence of $Q^{ab}_{(0)}$ is given by

$$2\nabla_b Q^{ab}_{(0)} = \theta^a - L\xi^a - 2E^{ab}\xi_b, \tag{A7}$$

where E^{ab} is the variational derivative of L, the symplectic current reads

$$\begin{aligned}\theta^a &= Z^{a:bcde}\mathcal{L}_\xi R_{bcde} \\ &+ 2X^{abcd}_{(0)}\left(\nabla_b\nabla_c\xi_d - R^f{}_{bcd}\,\xi_f\right) \\ &+ \left(4\nabla_d X^{abcd}_{(0)} + A^{abc}\right)\nabla_{(b}\xi_{c)},\end{aligned} \tag{A8}$$

and A^{abc} is a combination of terms of the form ZR. If the entropy density (A6) is to give rise to the field equation derived from L, the divergence of $Q^{ab}_{(1)}$ must take the form

$$\begin{aligned}2\nabla_b Q^{ab}_{(1)} &= -Z^{a:bcde}\mathcal{L}_\xi R_{bcde} + F\xi^a \\ &+ F^{abcd}\left(\nabla_b\nabla_c\xi_d - R^f{}_{bcd}\,\xi_f\right) \\ &+ F^{abc}\nabla_{(b}\xi_{c)},\end{aligned} \tag{A9}$$

for some tensors F. From the symmetries of the Riemann tensor and the definition of the Lie derivative, this can be seen to be equivalent to the system

$$\begin{aligned}W^{a[bc]}_{(1)} + \nabla_d X^{adbc}_{(1)} &= -4Z^{a:[b}{}_{def}R^{c]def}, \\ \nabla_b W^{abc}_{(1)} + X^{abde}_{(1)}R^c{}_{bde} &= -Z^{a:bdef}\,\nabla^c R^b{}_{def} + Fg^{ac}\end{aligned} \tag{A10}$$

In conclusion, if entropy densities of the desired form exist, there must exist an integrating function F such that (A10) has a solution. There is a relation between the integrating function F and the integrating function Φ of Eqn. (47). Namely, comparing (A7)-(A9) with (44) reveals that in this case $\Phi = L/2 + F/2$.

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